COMBINING FUNCTIONAL DECOMPOSITIONS WITH T-GATES IMPLEMENTATIONS FOR TERNARY COMBINATIONAL NETWORKS

Oleg Cernian

University of Craiova,
Computer and Communications Engineering Department
13, Alexandru Ioan Cuza Street, Craiova

Abstract: The subject of multiple valued logic is gaining more and more interest due to the gradual approaching of the physical limits of the binary logic (Hurst, 1984). Having in view the advantages of information representation the ternary logic is occupying a central position of reserach. In this paper it is presented a combination between the simple multiple disjunctive decomposition of ternary logic functions and implementations with ternary T-gates. A comparison of complexity of implementations is carried out based on their costs.

Keywords: functional decomposition, ternary T-gates, ternary switching functions, decomposition matrix, cost of CLNs.

1. SIMPLE-MULTIPLE FUNCTIONAL DECOMPOSITION OF TERNARY LOGIC FUNCTIONS

Let $f(x_1 \ldots x_n)$ be a ternary logic function, defined on the set of ternary variables

$$X = \{x_1 \ldots x_n\}$$

On this set it is defined a partition *Y*|*Z* where $Y = \{x_{i_1} \ldots x_{i_p}\}$ and $Z = \{x_{i_{p+1}} \ldots x_{i_n}\}$

If the function f can be decomposed as follows:

$$f(x_1 \ldots x_n) = g_3[g_2(Y), g_1(Y), Z] \qquad (1)$$

where: $Y \bigcup Z = X, Y \bigcap Z = \phi$

g_3, g_2 and g_1 are ternary logic functions, then it is said that the function $f(x_1 \ldots x_n)$ allows a simple multiple disjunctive functional decomposition of the second order.

In order to avoid trivial decompositions it is imposed the condition *card* $\{Y\} > 1$.

Such a decomposition is called simple, because the set of variables is split into two subsets Y and Z, and it is called multiple, because there exist at least two subsets g_2 and g_1 that depend on the subset of variables Y.

The subset $Y = \{x_{i_1} \ldots x_{i_p}\}$ is called the bound set of variables, whereas the subset $Z = \{x_{i_{p+1}} \ldots x_{i_n}\}$ is called the free set of variables.

The principle of the second order simple multiple disjunctive functional decomposition is depicted in a graphical form in Fig 1.

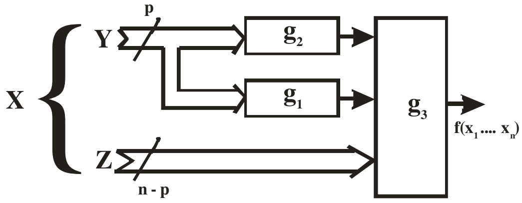

Fig. 1. Graphical representation of the second order simple multiple disjunctive decomposition.

The generalization is straightforward for an (*m*-1) order simple-multiple disjunctive functional decomposition of the ternary logic function $f(x_1 \ldots x_n)$:

$$f(x_1 \ldots x_n) = g_m \left[g_{m-1}(Y), g_{m-2}(Y), \ldots g_1(Y), Z \right] \quad (2)$$

where $Y \bigcup Z = X, Y \bigcap Z = \phi$

$g_1, g_2 \ldots g_m$ are ternary logic functions.

The principle of the (*m-1*) order simple multiple functional decomposition of the logic function $f(x_1 \ldots x_n)$ is depicted in Fig 2.

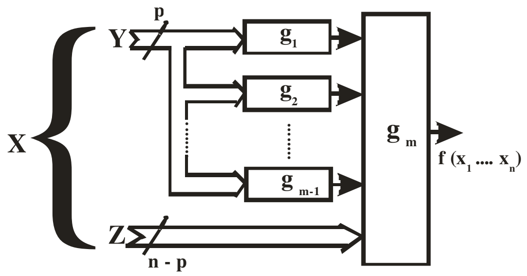

Fig. 2. The (*m-1*) order simple multiple disjunctive functional decomposition

The advantage of using functional decompositions for ternary function implementation is obvious, since it is derived a drastic reduction of the implemented functions: instead of realizing a single *n* variable ternary function, there are synthesized (*m*-1) ternary functions depending on *p* variables and one ternary function depending on *n-p* +*m-1* variables.

The basic tool used for analysis of decomposition property of switching functions is the decomposition matrix (Ashunhurst, 1957), which can be easily extended to ternary functions.

In particular, the order of multiplicity of columns is designated by *q* and if $q \leq 3$ then the function allows a simple disjunctive functional decomposition (or simple multiple of order 1):

$$f(x_1 \ldots x_n) = g_2[g_1(Y), Z] \quad (3)$$

where $Y \bigcup Z = X, Y \bigcap Z = \phi$

If *q*>3 then the analysis is continued for identifying a multiple functional decomposition. If $3 < \mathrm{q} \leq 9$ then the ternary function allows a second order simple multiple functional decomposition:

$$f(x_1 \ldots x_n) = g_3[g_2(Y), g_1(Y), Z] \quad (4)$$

In general, if $3^{m-2} < \mathrm{q} \leq 3^{m-1}$ then the ternary function allows an *(m-1)* order simple multiple functional decomposition:

$$f(x_1 \ldots x_n) = g_m \left[g_{m-1}(Y), g_{m-2}(Y), \ldots g_1(Y), Z \right] \quad (5)$$

The effective decomposition is done in a twofold procedure:

a) calculation of the function $g_m(g_{m-1} \ldots g_1, Z) = g_m(g_{m-1} \ldots g_1, x_{i_{p+1}} \ldots x_{i_n})$.

b) calculation of the subfunctions $g_{m-1}(Y) \ldots g_1(Y)$.

By considering the simplest case, that referring to the second order simple multiple functional decomposition, this procedure is implemented as follows:

1) For the partition *Y*|*Z* it is constructed the matrix 2^{n-p}x2^p – dimensional, called the decomposition matrix.
2) It is determined the column multiplicity and it is assumed that it is 9; the reduced decomposition matrix will contain 9 columns designed $C_0C_1C_2C_3C_4C_5C_6C_7C_8$
3) The columns are associated to the ternary variables g_1 and g_2 allowing the following set of assignments: {00, 01, 02, 10, 11, 12, 20, 21, 22}.
4) These 9 combinations are put in correspondence with columns $C_0C_1 \ldots C_8$. Obviously, there can be defined 9! associations; it is considered one of these associations.
5) It is reconstructed the reduced decomposition matrix as a ternary diagram of (*n-p*+2) variables (Fig 3)

	Z \ g_2g_1	00	01	02	10	11	12	20	21	22
3^{n-p} rows	0 00									
	0 01									
	0 02									
	⋮									
	2 22									

Fig. 3. Reduced decomposition matrix

6) It is synthesized directly in the minimal form the ternary function $g_3\left[g_2 g_1 x_{i_{p+1}} \ldots x_{i_n}\right]$
7) It is constructed a new ternary diagram containing a single row and 2^p columns by placing in the column C_i the union of two ternary values {*r, t*} that correspond to the adopted assignment in step 4. The ternary value *r* corresponds to the function $g_2(x_1 \ldots x_p)$, while the ternary value *t* corresponds to the function $g_1(x_1 \ldots x_p)$.

8) The ternary functions $g_1(x_1 \ldots x_p)$ and $g_2(x_1 \ldots x_p)$ are directly synthesized in the minimal form.

2. T-GATE TERNARY FUNCTION IMPLEMENTATIONS BASED ON THE SIMPLE MULTIPLE FUNCTIONAL DECOMPOSITION

According to (Lee and Chen, 1956) any ternary logic function of n variables can be implemented by a tree structure of T-gates containing $\frac{3^n - 1}{2}$ T gates. Each ternary T gate has four inputs and one output (Fig 4):

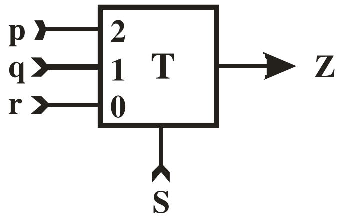

Fig. 4. A basic ternary T gate

The input S is called the control variable or selection variable, while the input-output relation is:

$$T(p,q,r;s) = p \cdot J_2(s) \bigcup q \cdot J_1(s) \bigcup r \cdot J_0(s) \tag{6}$$

where: p, q, r, s are ternary variables belonging to {0, 1, 2}

$$J_k(s) = \begin{cases} 2 & if \quad s = k \\ 0 & if \quad s \neq k \end{cases} \tag{7}$$

$\bigcup$ is the Max operator,

$\bigcap$ is the Min operator

A k – multiple control ternary T-gate [Higuchi et al., 1985] is depicted in Fig 5:

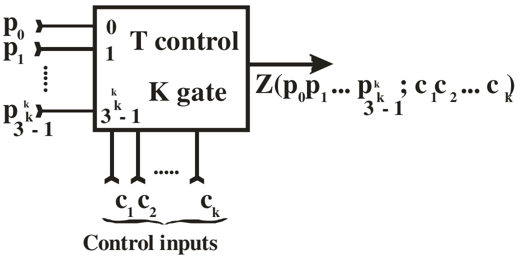

Fig. 5. K multiple control ternary T gate

where $C_1 \ldots C_k$ are k ternary control variables
$p_0p_1 \ldots p_{3^k-1}$ are 3^k residual ternary functions

The output Z is equal to residual function p_j, where $j = \sum_{i=1}^{k} C_i 3^{i-1}$ (8)

j is a scalar defining the index of the input transmitted to the output of the gate.

In case of k – multiple control T gates the canonical implementations of a ternary function $f(x_1 \ldots x_n)$ depend of the values of n and k, having $l = \left\lceil \frac{n}{k} \right\rceil$ levels and a total of $\frac{3^{\left\lceil \frac{n}{k} \right\rceil \cdot k} - 1}{3^k - 1}$ T gates.

Obviously, for large values of n the number of T gates is becoming inconvenient for canonical implementations.

Therefore, techniques for reducing the number of required T gates for implementations of ternary functions were proposed. The target of such techniques is focusing on the number of levels and of gates in a level, with a direct impact on the cost of the implementation.

It is proposed a method based on the functional decomposition of ternary logic functions. In particular, the considered decomposition corresponds to the simple multiple disjunctive decomposition presented in paragraph 1. The set of ternary variables $X = \{x_1 \ldots x_n\}$ is split into two subsets $Y = \{x_{i_1} \ldots x_{i_p}\}$ and $Z = \{x_{i_{p+1}} \ldots x_{i_n}\}$, $Y \bigcup Z = X, Y \bigcap Z = \phi$.

It is analyzed if $f(x_1 \ldots x_n)$ is decomposable according to the given formula for an (m-1) order simple multiple disjunctive functional decomposition:

$$f(Y,Z) = g_m\left[g_{m-1}(Y), g_{m-2}(Y), \ldots g_1(Y), Z\right]$$

Then, instead of synthesizing a ternary function of n variables, it has to be synthesized (m-1) ternary functions of p variables and one of (n-p+m-1) variables.

An analysis of the costs of implementations follows, by considering both basic ternary T-gates and multiple control ternary T-gates.

It is assumed that the cost of a ternary T-gate is $(T).

Then, for the canonical implementation of the function $f(x_1 \ldots x_n)$ the costs are:

$$M_c = \left(\frac{3^{\left\lceil \frac{n}{k} \right\rceil} - 1}{3^k - 1} \right) \cdot \$(T) \tag{9}$$

in case of k - multiple control ternary T gates, and

$$M_c = \left(\frac{3^n - 1}{2} \right) \cdot \$(T) \qquad (10)$$

in case of basic ternary T gates.

According to the presented functional decomposition, the cost of implementation is defined by two components: those (m-1) subfunctions $g_1(Y)\ldots g_{m-1}(Y)$ and the function $g_m(g_1,\ldots g_{m-1},Z)$.

Hence,

$$M_{DES} = (m-1)\cdot M_p + M_{m+n-p-1} =$$

$$(m-1)\left(\frac{3^{\left\lceil \frac{p}{k} \right\rceil} - 1}{3^k - 1} \right) \cdot \$(T) +$$

$$\left(\frac{3^{\left\lceil \frac{m+n-p-1}{k} \right\rceil \cdot k} - 1}{3^k - 1} \right) \cdot \$(T) \qquad (11)$$

in case of k-multiple control ternary T gates, and

$$M_{DES} = (m-1)\cdot M_p + M_{m+n-p-1} =$$

$$(m-1)\left(\frac{3^p - 1}{2} \right) \cdot \$(T) + \left(\frac{3^{n-p+m-1} - 1}{2} \right) \cdot \$(T) \qquad (12)$$

in case of basic ternary T gates.

It must be checked the inequality $M_{DES} < M_c$, which will guarantee the efficiency of the decomposed solution.

3. EXAMPLES OF USING BASIC TERNARY T-GATES

I. It is given the following ternary logic function depending on variables $\{x_1 x_2 x_3\}$:

$$f(x_1 x_2 x_3) = \begin{cases} 2 : 2,4,6,10,12,17,18,23,25 \\ 1 : 1,3,8,9,14,16,20,22,24 \\ 0 : 0,5,7,11,13,15,19,21,26 \end{cases}$$

On the set X it is defined the partition $x_1 x_2 / x_3$, therefore $Y=\{x_1 x_2\}$ and $Z=\{x_3\}$.

The required condition $p \geq 2$ is verified. It is constructed the decomposition matrix (Fig, 6)

x_3 \ $x_1 x_2$	00	01	02	10	11	12	20	21	22
0	0	1	2	1	2	0	2	0	1
1	1	2	0	2	0	1	0	1	2
2	2	0	1	0	1	2	1	2	0
Column identifier	A	B	C	B	C	A	C	A	B

Fig. 6. Decomposition matrix

The column multiplicity is 3, therefore a simple disjunctive functional decomposition is possible, according to the decomposition formula:

$$f(x_1 x_2 x_3) = g_2[g_1(x_1 x_2), x_3] \qquad (13)$$

The distinct columns are:

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

These columns are associated to ternary logic values {0,1,2} associated to function $g_1(x_1 x_2)$:

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \to 0, \quad \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \to 1, \quad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \to 2$$

The reduced partition matrix is used for synthesis of the function $g_2(g_1 x_3)$:

x_3 \ g_1	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Fig. 7. Reduced decomposition matrix

Synthesis of g_2 subfunction according to T canonical form (Thelliez, 1971):

$$g_2(g_1 x_3) = T[T(0,1,2;x_3), T(1,2,0;x_3), T(2,0,1;x_3); g_1] \qquad (14)$$

It is constructed the table for $g_1(x_1 x_2)$ synthesis:

x_1 / x_2	0 / 0	0 / 1	0 / 2	1 / 0	1 / 1	1 / 2	2 / 0	2 / 1	2 / 2
$g_1(x_1 x_2)$	0	1	2	1	2	0	2	0	1

Fig. 8. Truth table for synthesis of $g_1(x_1 x_2)$

$$g_1(x_1 x_2) = T[T(0,1,2;x_1), T(1,2,0;x_1), T(2,0,1;x_1); x_2] \qquad (15)$$

Implementation of the decomposed structure for $f(x_1x_2x_3)$ is presented in fig. 9.

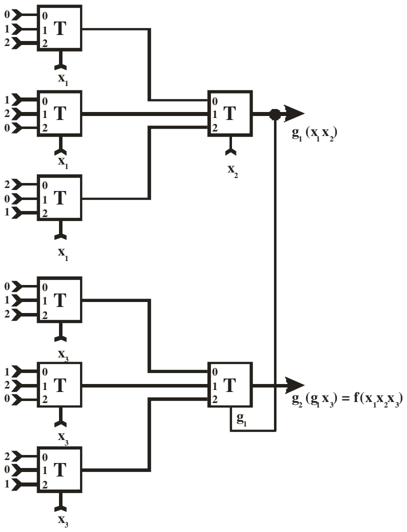

Fig. 9. Implementation of the ternary function $f(x_1x_2x_3)$

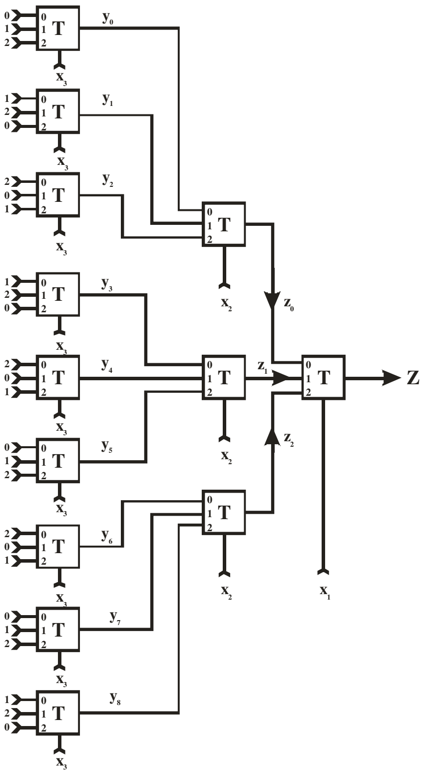

Fig. 10 Canonical implementation of the function f

By synthesizing the ternary function in the T - canonical form the following equation is derived:

$$f(x_1x_2x_3) =$$
$$[T[[T[0,1,2;x_3],T[1,2,0;x_3],T[2,0,1;x_3];x_2],$$
$$[T[1,2,0;x_3],T[2,0,1;x_3],T[0,1,2;x_3];x_2],$$
$$[T[2,0,1;x_3],T[0,1,2;x_3],$$
$$T[1,2,0;x_3];x_2]];x_3] \qquad (16)$$

Canonical implementation would require 13 basic T gates (Fig. 10).

Evaluation of costs of implementation:
Canonical solution:

$$M_c = \frac{3^3-1}{2}\cdot \$(T) = 13\$(T)$$

Decomposed solution:

$$M_{DES} = (1\cdot M_2 + M_{1+3-2})\$(T) = (M_2 + M_2)\$(T) =$$

$$2M_2\$(T) = 2\cdot\frac{3^2-1}{2}\cdot\$(T) = 8\$(T)$$

Therefore, the partition x_1x_2/x_3 is accepted, and the decomposition validated, as yielding a cheaper solution.
It was ignored further optimization of the derived T-structures, allowing more economical implementations (Higuchi et al., 1985)

II. It is given the following ternary logic function depending on variables $\{x_1x_2x_3\, x_4\}$:

$$h(x_1x_2x_3x_4) = \begin{cases} 2: 0,1,2,5,10,15,17,25 \\ 1: 3,6,12,13,14,19,20,26 \\ 0: 4,7,8,9,11,16,18,21,22,23,24 \end{cases}$$

On the set X it is defined the following partition $x_1x_2x_3/x_4$, thus $Y=\{x_1x_2\, x_3\}$ and $Z=\{x_4\}$.
It is constructed the decomposition matrix (Fig. 11)

x_1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
x_2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
x_3	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
x_4																											
0	2	1	1	1	1	0	0	1	1	1	1	2	1	2	1	1	1	1	0	2	0	1	0	0	0	0	0
1	2	0	0	0	0	2	2	1	0	0	1	0	0	0	0	0	1	0	1	0	1	0	1	2	2	0	2
2	2	2	0	2	0	0	0	1	2	2	1	2	2	2	0	2	1	0	1	2	1	0	1	0	0	0	1
Column identifier	I	A	B	A	B	C	C	D	A	A	D	E	A	E	B	A	D	B	F	E	F	B	F	C	C	G	H

Fig. 11. The decomposition matrix for $h(x_1x_2x_3x_4)$

The column multiplicity is 9, therefore it is allowed a simple multiple disjunctive decomposition of the second order, according to the presented formula:

$$h(x_1x_2x_3x_4) = g_3\left[g_1(x_1x_2x_3), g_2(x_1x_2x_3), x_4\right]$$

The distinct columns are associated to the ternary combinations of variables g_1 and g_2:

$$\begin{pmatrix}2\\2\\2\end{pmatrix} \to 00, \quad \begin{pmatrix}1\\0\\2\end{pmatrix} \to 01, \quad \begin{pmatrix}1\\0\\0\end{pmatrix} \to 02,$$

$$\begin{pmatrix}0\\2\\0\end{pmatrix} \to 10, \quad \begin{pmatrix}1\\1\\1\end{pmatrix} \to 11, \quad \begin{pmatrix}2\\0\\2\end{pmatrix} \to 12,$$

$$\begin{pmatrix}0\\1\\1\end{pmatrix} \to 20, \quad \begin{pmatrix}0\\0\\0\end{pmatrix} \to 21, \quad \begin{pmatrix}0\\2\\1\end{pmatrix} \to 22$$

The reduced decomposition matrix is depicted in Fig. 12

g_1g_2 / x_4	00	01	02	10	11	12	20	21	22
0	2	1	1	0	1	2	0	0	0
1	2	0	0	2	1	0	1	0	2
2	2	2	0	0	1	2	1	0	1

Fig. 12. The reduced decomposition matrix

Specification of the subfunction g_3

$$g_3(g_1g_2g_3) = \begin{cases} 2: 0,1,2,10,5,15,17,25 \\ 1: 3,6,12,13,14,19,20,26 \\ 0: 4,7,8,9,11,16,18,21,22,23,24 \end{cases}$$

Synthesis of the subfunction $g_3(g_1,g_2,x_4)$ with basic T gates is presented in Fig. 13

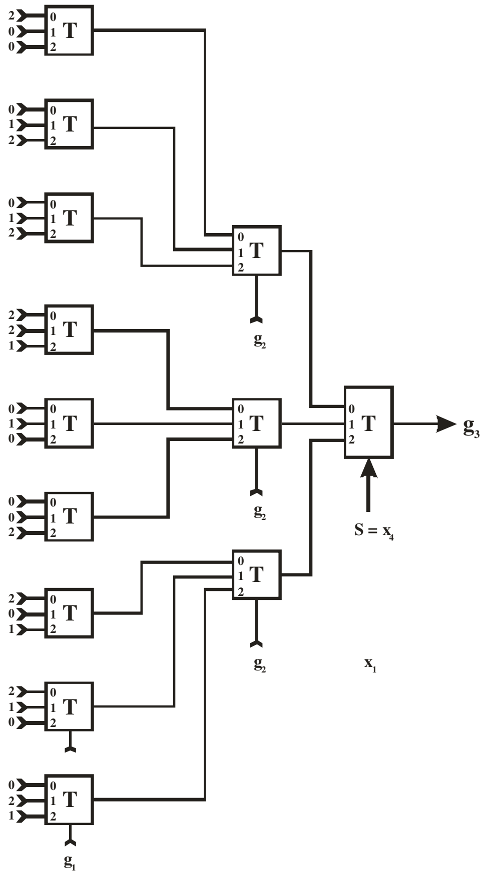

Fig. 13. Implementation of the subfunction $g3$ with basic T gates

The sysnthesis of the subfunctions $g1$ and $g2$ is carried out from the truth tables given in Fig. 14.

The implementation of the subfunctions $g_1(x_1x_2x_3)$ and $g_2(x_1x_2x_3)$ is presented in figures 15 and 16.

x_1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
x_2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
x_3	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
g_1	0	1	2	1	2	0	0	1	1	1	1	2	1	2	2	1	1	2	0	2	0	2	0	0	0	1	2
g_2	0	0	0	0	0	1	1	1	0	0	1	1	0	1	0	0	1	0	2	1	2	0	2	1	1	2	2

Fig. 14. Truth tables for synthesis of subfunctions g_1 and g_2

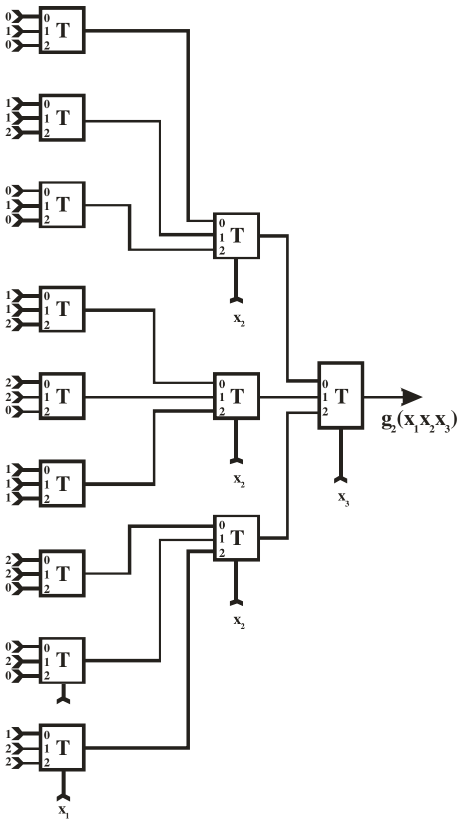

Fig. 15. Implementation of the subfunction g_2

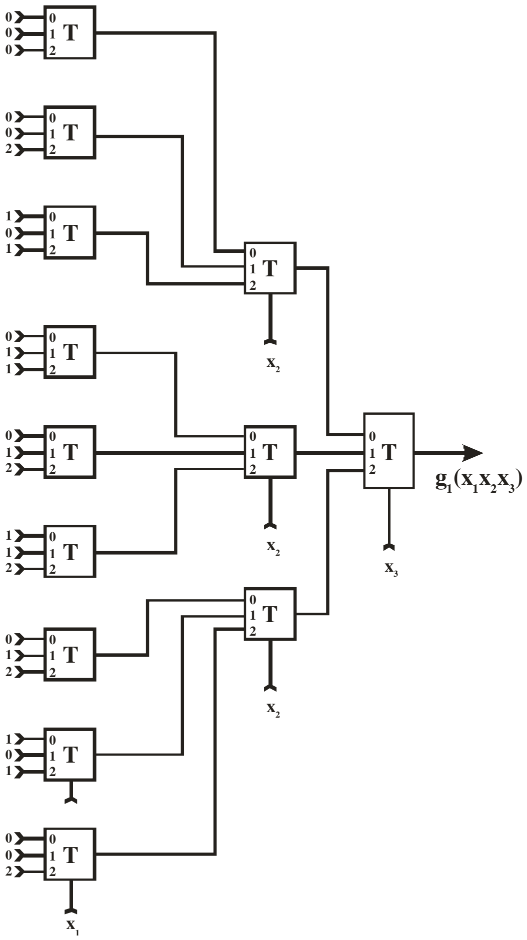

Fig. 16. Implementation of the subfunction g_1

By assembling the constructed subfunctions it is obtained eventually the entire structure for $h(x_1x_2x_3x_4)$ (Fig. 17)

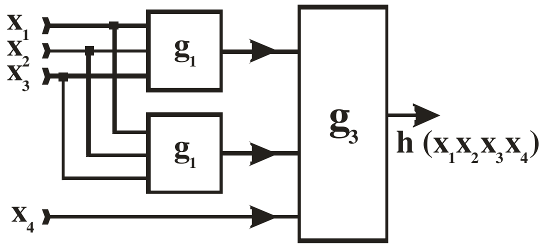

Fig. 17. Implementation of the function $h(x_1x_2x_3x_4)$

Since $M_{DES} = 39 < M_C = 40$ the decomposition is validated.

REFERENCES

Ashunhurst, R.L., (1957) The Decomposition of Switching Circuits, *Proc ISTS*

Lee, C.Y.,Chen,W.H., (1956) Several Valued Combinational Switching Circuits,*Trans AIEE,* **vol 75**

Bartacco, V., Damiani, M. (1997) The Disjunctive Decomposition of Logic Functions, *ICCAD*

Higuchi, T., Kameyama, M., (1977) Synthesis of M-V Networks Based on Tree Type Universal Logic Modules, *IEEE Trans. Comp,* **C-26**.

Higuchi, T., Kameyama, M., (1985) Ternary Logic System Based on the T-gates, *Proc. of the 15th Int.Symp. on MVL*

Hurst, S.L., (1984) Multiple-Valued Logic-Its Status and its Future, *IEEE Trans Comp,* **C30**.

Porat, D.I., (1969) Three Valued Digital Systems, *Proc IEE,* **vol 110**.

Thelliez, S., (1971) *Introduction a l'Etude des Structures Ternnaires de Commutation*, Gordon & Breach